Common patterns for order and metric fixed point theorems

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There are plenty
of reasons why
we can forget
the distinction between
order and metric
fixpoint theorems.

(The usual suspects: A. Einstein or M. Twain)

Order vs. metric fixpoints

(Knaster-Tarski) An order-preserving map on a complete lattice has the least and the greatest fixed point.

(Banach) A contraction on a complete metric space has a unique fixed point.

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OUR GOAL: Show that both are instances of a single theorem with a constructive proof.

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Proof idea: Iterate f:

$$\bot, f(\bot), f^2(\bot), f^3(\bot), \ldots$$

and eventually you will reach the least fixed point. Flip the lattice to get the greatest one.

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(Banach) A contraction $f\colon X \to X$ on a complete metric space has a unique fixed point.

Proof idea: Iterate f:

$$x, f(x), f^2(x), f^3(x), \dots$$

and no matter what $x \in X$ you started with, eventually you will reach the same fixed point.

Unification

(Lawvere 1973) Orders and metric spaces are instances of quantale-enriched categories.

(Edalat & Heckmann 1998) A topology of a complete metric space is homeomorphic to a subspace Scott topology on maximal elements of a continuous directed-complete partial order.

Unification a la Lawvere

A bit of cleaning first!

A metric on a set X:

$$d_X : X \times X \to [0, \infty)$$

We use it as:

$$d_X(x, y), d_X(y, z), \ldots$$

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SYMMETRY IS NOT TOO IMPORTANT!

A metric on a set X:

$$X\colon X \times X \to [0, \infty]$$

$$X(x, y) = 0 \text{ iff } \quad x = y$$

$$X(x, y) = X(y, x)$$

$$X(x, y) \leqslant X(x, z) + X(z, y)$$

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CONCLUSION: $\leqslant_X$ is a partial order.

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BETTER CONCLUSION:

Replace $[0,\infty]$ by $\{0,\infty\}$ to switch from metrics to orders.
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Unification a la Lawvere: the setup

Let $\mathcal{Q}$ be a complete lattice with $+$ and 0.

A $\mathcal{Q}$-category is a set X with a structure $X : X \times X \to \mathcal{Q}$ satisfying:

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For $\mathcal{Q} = \mathbf{2}$ we recover partial orders.
For $\mathcal{Q} = [0, \infty]$ we recover metric spaces.
But other choices of $\mathcal{Q}$ are possible too.

More on the setup

A $\mathcal{Q}$-functor between $\mathcal{Q}$-categories is a function $f\colon X \to Y$ satisfying:

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2-functors are order-preserving maps.
$[0, \infty]$-functors are non-expansive maps between metric spaces.

$\mathcal{Q}$-functors of type $X \to Y$ form a $\mathcal{Q}$-category when considered with the structure:

$$Y^X(f, g) := \sup_{x \in X} Y(fx, gx).$$

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Consider a net $(x_i)_{i \in I}$ such that

> *from some N onwards, elements of the net are arbitrarily close to each other.*

For $\mathcal{Q} = \mathbf{2}$, $(x_i)_{i \in I}$ is eventually a directed set.
For $\mathcal{Q} = [0, \infty]$, $(x_i)_{i \in I}$ is a Cauchy net.

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DEFINITION: An *ideal* on X is a map:

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FACT: Ideals on X form a $\mathcal{Q}$-category:

$$\mathbb{I}X(\phi, \psi) := \sup_{x \in X} (\psi x - \phi x).$$

Last slide about the setup

DEFINITION: A $\mathcal{Q}$-category X is $\mathbb{I}$-complete if there exists a map $\mathcal{S}\colon \mathbb{I}X \to X$ with

$$X(\mathcal{S}\phi, x) = \mathbb{I}X(\phi, X(-, x))$$

for all $\phi \in \mathbb{I}X$ and $x \in X$.

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IMPORTANT:
Replacing $\mathbb{I}$ by $\widehat{(\cdot)}$ we have a notion of $\widehat{(\cdot)}$-completeness.
Replacing $\mathbb{I}$ by any suitable J we have a notion of J-completeness.

What we gained

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Still we have other choices of J and $\mathcal{Q}$!

Fixpoints again

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OUR GOAL: Show that both are instances of a single theorem with a constructive proof.

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BOTH FOLLOW FROM: A $\mathcal{Q}$-functor $f : X \to X$ on a J-complete $\mathcal{Q}$-category has a fixed point, providing the direct image $\mathcal{Q}$-functor

$$f^* : JX \to JX$$

$$f^*(\phi) := \inf_{z \in X} (\phi(z) + X(-, fz))$$

has a fixed point.

Proof idea

THEOREM A $\mathcal{Q}$-functor $f\colon X \to X$ on a J-complete $\mathcal{Q}$-category has a fixed point, providing that $f^*\colon JX \to JX$ has a fixed point ϕ.

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2. f is a $\mathcal{Q}$-functor implies f is $\leqslant_X$-preserving.
3. f^* has a fixpoint ϕ, implies $\mathcal{S}\phi = \mathcal{S}f^*(\phi) \leqslant_X f(\mathcal{S}\phi)$.
4. Then we use Pataraia's proof of the fact that an order-preserving map on a dcpo has a least fixed point. QED.

How to obtain classic fixed point theorems

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More fixpoints

(Bourbaki-Witt) An expanding map $f\colon X \to X$ on a dcpo X has a fixed point.

(James Caristi, 1976) Let $f\colon X \to X$ be an arbitrary map on a complete metric space. If there exists a l.s.c. map $\varphi\colon X \to [0, \infty)$ such that:

$$(*) \qquad X(x, fx) + \varphi(fx) \leqslant \phi(x),$$

then f has a fixed point.

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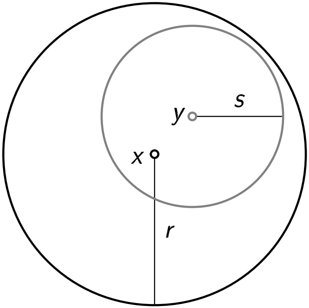
OUR GOAL: Show that both are instances of a single theorem that can have no constructive proof.

Unification a la Edalat & Heckmann

Edalat, A. and Heckmann, R. (1998) A computational model for metric spaces. *Theoretical Computer Science* **193**(1–2), pp. 53–73.



$$\mathbf{B}X := \{\langle x, r\rangle \mid x \in X \text{ and } r \geqslant 0\} \subseteq X \times \mathbb{R}_+$$

$$\langle x, r\rangle \leqslant \langle y, s\rangle \quad \text{iff} \quad X(x, y) + s \leqslant r$$

$$X \cong \{\langle x, 0\rangle \mid x \in X\}(= \max(\mathbf{B}X) \text{ providing } X \text{ is } T_1).$$

Unification a la Edalat & Heckmann

Edalat and Heckmann's construction works the same for $\mathcal{Q}$-categories. Therefore:

THEOREM
X is an $\mathbb{I}$-complete $\mathcal{Q}$-category iff $(\mathbf{B}X, \leqslant)$ is a dcpo.

Analysis of Caristi's Theorem

(Nonsymmetric Caristi) Let $f: X \to X$ be an arbitrary map on a $\mathbb{I}$-complete $[0, \infty]$-category. If there exists a l.s.c. map $\varphi: X \to [0, \infty)$ such that:

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6. Hence (Nonsymmetric Caristi) has no constructive proof either.

But...

... maybe (Caristi) has a constructive proof?

NO.
The proof idea is due to Hannes Diener.

Hannes Diener (photo by Andrej Bauer)

Hannes' proof

Let $a, b \in \mathbb{R}$ be We will show that (Caristi) implies that for any two non-negative reals a, b such that $\neg(a \neq 0 \wedge b \neq 0)$, we have either $a = 0$ or $b = 0$.

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2. I have argued that theorems of Bourbaki-Witt and Caristi are in essence 'the same' — by switching from a metric space X to its formal ball model $\mathbf{B}X$.
3. In fact, (Nonsymmetric Caristi) can be further generalized to become a source theorem for both classic results mentioned in 2.

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6. Hence $f(\top(\perp)) = \top(\perp)$, and for any other fixpoint $x \in X$, the set $\downarrow x$ satisfies (a)-(c), and thus $\top(\perp) \in C \subseteq \downarrow x$. QED.