RECONCILIATION OF ELEMENTARY ORDER AND METRIC FIXPOINT THEOREMS

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Abstract. We prove two new fixed point theorems for generalized metric spaces and show that various fundamental fixed point principles, including: Banach Contraction Principle, Caristi fixed point theorem for metric spaces, Knaster-Tarski fixed point theorem for complete lattices, and the Bourbaki-Witt fixed point theorem for directed-complete orders, follow as corollaries of our results.

1. Introduction

In 1973 William Lawvere [11] observed that both orders and metrics are examples of quantale-enriched categories. Indeed, a partial order can be viewed as a category enriched in the two-element boolean algebra, and a metric space is a category enriched in extended non-negative real numbers $[0, \infty]$ with addition. Lawere's ideas proved to be extremely influential over the years and recently we have been witnessing a birth of a unified algebraic theory of some of the most fundamental structures in mathematics: orders, metrics, topologies and uniformites [6, 10].

A futher impulse to study partial order and metric spaces from a unified perspective comes from theoretical computer science, where both structures are used as mathematical models for denotational semantics of programming languages [1, 3, 7]. As it happens one of the main reasons for usefulness of these structures in semantics is existence of fixed points. For a concrete example: in Scott-type denotational semantics programs are interpreted as certain continuous functions and *while* loops are then interpreted as fixed points of these functions.

It is the purpose of this paper to prove some of the most fundamental fixed point principles from both metric and order fixed point theory in the uniform mathematical framework offered by Lawvere's research programme. We prove that there is a single fixed point theorem for generalized metric spaces that has both Banach Contraction Principle [2] and Knaster-Tarski fixed point theorem [14] as corollaries. Likewise, we demonstrate that the Caristi fixed point theorem for complete metric spaces [5] and the Bourbaki-Witt fixed point theorem for directed-complete orders [4, 17] are both instances of a single fixed point theorem for generalized metric spaces.

This article is an expanded and corrected version of a conference publication [16].

2. The setup

In order to make our work accessible to the widest spectrum of readers we refrain from using the specialised language of category theory and phrase our results in the nomenclature of metrics and orders.

2010 *Mathematics Subject Classification.* 06A75, 54H25, 54E50.

This work was supported by grant number N206 3761 37 funded by Polish Ministry of Science and Higher Education.

2.1. **Distance spaces.** Our primary object of investigation is a set X together with a distance into $[0,\infty]$ that satisfies: (a) $X(x,y) = X(y,x) = 0$ iff $x = y$, (b) $X(x,z) \leqslant X(x,y) + X(y,z)$, for all $x,y,z \in X$. Note that symmetry of the distance does not follow from the axioms, hence every X has its nontrivial *dual*, $X^{op}(x,y) := X(y,x)$. On the other hand, adding the axiom of symmetry, together with $X(x,y) < \infty$ for all $x,y \in X$, makes X a metric space. Although a traditional name for such spaces is: (extended) quasi-metric spaces [8], in what follows, for the sake of simplicity, we call X a distance space.

Example 2.1. We define $c \dot{-} b := \inf\{a \mid c \leqslant a + b\}$ for $b,c \in [0,\infty]$ and observe that it is in fact the usual substraction truncated at zero. Then $[0,\infty]$ itself is a distance space with $[0,\infty](b,c) := c \dot{-} b$.

Example 2.2. If $(X,\leqslant)$ is a partial order, then it can be turned into a distance space by declaring $X(x,y) = 0$ iff $x \leqslant y$ and $X(x,y) = \infty$ otherwise.

2.2. **Induced order.** Any distance space carries an intrinsic partial order: $x \leqslant_X y$ iff $X(x,y) = 0$. In metric spaces the induced order reduces to equality. In any poset $(X,\leqslant)$, considered as a distance space, we have $\leqslant_X = \leqslant$.

2.3. **Maps between distance spaces.** A map $f\colon X \to Y$ is *non-expansive* if $X(x,y) \geqslant Y(fx,fy)$ for all $x,y \in X$. A crucial role in this paper is played by the set $\widehat{X}$ of all non-expansive maps of type $X^{op} \to [0,\infty]$. By triangle inequality, for any $x \in X$, the map $\mathsf{y}_X(x)(y) := X(y,x)$ is an element of $\widehat{X}$. For other elements of $\widehat{X}$ we will use Greek letters $\phi, \psi, \ldots$, etc.

The set $\widehat{X}$ becomes a distance space with $\widehat{X}(\phi,\psi) := \sup_{z\in X}(\psi z \dot{-} \phi z)$.

2.4. **The generalized direct image of a function.** Let $f\colon X \to Y$ be any non-expansive map. Consider $f^*\colon \widehat{X} \to \widehat{Y}$ given by

$$f^*(\phi)(y) := \inf_{x\in X}(\phi x + Y(y,fx)).$$

For example, if $z \in X$, then $f^*(\mathsf{y}_X(z))(y) = \inf_{x\in X}(X(x,z) + Y(y,fx)) = Y(y,fz)$ for all $y \in Y$. Hence $f^*(\mathsf{y}_X(z)) = \mathsf{y}_Y(fz)$.

Since $\widehat{X}(\phi,\psi) + f^*(\phi)(y) \geqslant \inf_{x\in X}(\psi x + Y(y,fx)) = f^*(\psi)(y)$, we get $\widehat{X}(\phi,\psi) \geqslant \widehat{Y}(f^*(\phi), f^*(\psi))$, i.e. f^* is non-expansive. The map f^* can be interpreted as a *generalized direct image* of f, see Sect. 2.6 of [15].

2.5. **Ideals.** Consider an operation J that assigns to every distance space X a subset JX of $\widehat{X}$ in such a way that each JX: (a) contains all $\mathsf{y}_X x$ for all $x \in X$, and (b) is closed under generalized images of non-expansive maps. We call elements of JX *ideals on* X, and we refer to J as the *class of ideals*. Observe that each JX with the distance inherited from $\widehat{X}$ is a distance space.

Example 2.3. $\widehat{X}$ is a class of ideals.

Example 2.4. Define $\mathbb{C} \subseteq \widehat{X}$ by $\phi \in \mathbb{C}X$ if and only if $\phi x := \inf_{i\in I}\sup_{j\geqslant i} X(x,x_j)$ for some *forward Cauchy net* $(x_i)_{i\in I}$. Recall from [3] that a net $(x_i)_{i\in I}$ is forward Cauchy if

$$\forall \varepsilon > 0\ \exists N \in I\ \forall m \geqslant n \geqslant N \quad (\varepsilon > X(x_n,x_m)).$$

We will often call the above defined ϕ the ideal *associated with* $(x_i)_{i\in I}$.

Then $\mathbb{C}$ is a class of ideals [7].

Definition 2.5. We say that $\phi \in \widehat{X}$ has a *supremum* $\mathsf{S}_X\phi$ if, for all $x \in X$,

$$X(\mathsf{S}_X\phi, x) = \widehat{X}(\phi, \mathsf{y}_X x). \tag{2.1}$$

Furthermore, X *is J-complete* if every $\phi \in JX$ has a supremum.

It is easy to check that in a J-complete distance space X, the map $\mathsf{S}_X\colon JX \to X$ is well-defined and non-expansive.

Observe that in any distance space X, for any $y \in X$, we have $\mathsf{S}_X\mathsf{y}_X y = y$, since $\widehat{X}(\mathsf{y}_X y, \mathsf{y}_X x) = \sup_{z\in X}(X(z,x) \dot{-} X(z,y)) = X(y,x) \dot{-} X(y,y) = X(y,x)$.

As a reward for all the above preparations we get the following fact: completeness of metric spaces, directed-completeness of posets and completeness of lattices are all instances of J-completeness for some appropriate choices of J:

Lemma 2.6. *A metric space X is complete iff it is $\mathbb{C}$-complete.*

Proof. Assume X is a complete metric space. Let $\phi = \inf_{i\in I}\sup_{j\geqslant i} X(-, x_j) \in \mathbb{C}X$. Let $\varepsilon_n = 1/(n+1)$ for $n \in \omega$. Define a map $\mu\colon \omega \to I$ recursively as follows:

$$\mu(n) = \begin{cases} N(\varepsilon_0) & \text{if } n = 0, \\ \max\{N(\varepsilon_n), \mu(n-1)\} & \text{if } n > 0, \end{cases}$$

where each $N(\varepsilon_n) \in I$ is chosen to be an index such that for all $i \geqslant j \geqslant N(\varepsilon_n)$ we have $X(x_i, x_j) < \varepsilon_n$. By construction, the sequence $(x_{\mu(n)})_{n\in\omega}$ is Cauchy. Let $g \in X$ be its limit. It is now easy to see that $\phi = \mathsf{y}_X g$ and hence $\mathsf{S}_X\phi = \mathsf{S}_X\mathsf{y}_X g = g$.

Conversely, any Cauchy sequence is forward Cauchy, and so defines an $\mathbb{C}$-ideal; by the paragraph above, the supremum of this ideal is the limit of the sequence. ☐

Lemma 2.7. *A partial order $(X, \leqslant)$ is directed-complete iff it is $\mathbb{C}$-complete.*

Proof. Recall that a lower closure of a subset A of the poset X is

$${\downarrow}A = \{z \in X \mid \exists a \in A\ (z \leqslant a)\}.$$

Instead of ${\downarrow}\{x\}$ we write ${\downarrow}x$.

Let D be a directed subset of X. Then $(x_d)_{d\in D}$, where $x_d := d$, is a forward Cauchy net. Let ϕ be the ideal associated with $(x_d)_{d\in D}$. Then in fact

$$\phi(z) = \begin{cases} 0 & \text{if } z \in {\downarrow}D, \\ \infty & \text{otherwise} \end{cases}$$

hence ϕ is a characteristic map of ${\downarrow}D$, and $\phi \in \mathbb{C}X$. From (2.1), for any $x \in X$, we get $\mathsf{S}_X\phi \leqslant x$ iff $X(\mathsf{S}_X\phi, x) = 0$ iff $\widehat{X}(\phi, \mathsf{y}_X x) = 0$ iff ${\downarrow}D \subseteq {\downarrow}x$ iff $D \subseteq {\downarrow}x$. This proves that $\mathsf{S}_X\phi$ is the least upper bound of D.

Conversely, let $\phi = \inf_{i\in I}\sup_{j\geqslant i} X(x, x_j)$. Since the net $(x_i)_{i\in I}$ is forward Cauchy, there exists an index $i_0 \in I$ such that $x_n \leqslant x_m$ for all $i_0 \leqslant n \leqslant m$, i.e. the subset $D := \{x_i \mid i \in I\}$ of X is eventually directed. Hence ${\downarrow}D$ is an order-ideal and ϕ is its characteristic function. By hypothesis $\bigvee{\downarrow}D$ exists, and so for any $x \in X$, we have $X(\bigvee{\downarrow}D, x) = 0$ iff $\bigvee{\downarrow}D \leqslant x$ iff ${\downarrow}D \subseteq {\downarrow}x$ iff $\widehat{X}(\phi, \mathsf{y}_X x) = 0$, which shows that $\bigvee{\downarrow}D$ is the supremum of ϕ. ☐

Lemma 2.8. *A partial order $(X, \leqslant)$ is a complete lattice iff it is $\widehat{(\cdot)}$-complete.*

Proof. Observe that $\phi \in \widehat{X}$ iff ϕ is a characteristic map of a lower subset of X. Analogously to the proof of Lemma 2.7 we can show that X is $\widehat{(\cdot)}$-complete iff every lower subset has an order-supermum, which in turn is equivalent to X being a complete lattice. □

2.6. **Admissible classes of ideals.** Let us call a class J of ideals is *admissible* if the intrinsic order $\leqslant_X$ of each J-complete X is directed-complete.

Example 2.9. $\mathbb{C}$ is admissible.

Proof. Suppose $(x_i)_{i\in I}$ is $\leqslant_X$-directed. Define $\phi := \inf_{i\in I}\sup_{j\geqslant i} X(-, x_j)$. Since X is $\mathbb{C}$-complete, the supremum $\mathsf{S}_X\phi \in X$ exists, and now will show that it is the least upper bound of $(x_i)_{i\in I}$.

Let $k \in I$. Firstly observe $\phi(x_k) \leqslant \sup_{j\geqslant k} X(x_k, x_j) = 0$. Therefore $0 = \phi(x_k) = \widehat{X}(\mathsf{y}_X x_k, \phi) \geqslant X(x_k, \mathsf{S}_X\phi)$, whence $x_k \leqslant_X S_X\phi$. On the other hand take any upper-bound $u \in X$ of $(x_i)_{i\in I}$. Then $0 = X(x_k, u)$ for all $k \in I$, and consequently $0 = \inf_k \sup_{j\geqslant k} \widehat{X}(\mathsf{y}_X x_k, \mathsf{y}_X u) = \widehat{X}(\phi, \mathsf{y}_X u) = X(\mathsf{S}_X\phi, u)$, i.e. $\mathsf{S}_X\phi \leqslant_X u$. □

Example 2.10. $\widehat{(\cdot)}$ is admissible. In fact, if X is $\widehat{(\cdot)}$-complete, then $(X, \leqslant_X)$ is a complete lattice.

Proof. It is easy to see that the pointwise order on $\widehat{X}$ is closed under arbitrary suprema (denoted here as $\bigvee_{\widehat{X}}$). Let $A \subseteq X$ and $a \in A$. Clearly, $\mathsf{y}_X a \leqslant_{\widehat{X}} \bigvee_{\widehat{X}} \mathsf{y}_X[A]$, and hence $a = S_X\mathsf{y}_X a \leqslant_X \mathsf{S}_X(\bigvee_{\widehat{X}} \mathsf{y}_X[A])$. Now, if $A \leqslant_X u$, then $\bigvee_{\widehat{X}} \mathsf{y}_X[A] \leqslant_{\widehat{X}} \mathsf{y}_X u$, and so $\mathsf{S}_X(\bigvee_{\widehat{X}} \mathsf{y}_X[A]) \leqslant_X \mathsf{S}_X\mathsf{y}_X u = u$. So far we have shown that X has all non-empty suprema. But the least element of X is given by $\mathsf{S}_X\bot_{\widehat{X}}$, as for all $x \in X$, we have $\bot_{\widehat{X}} \leqslant_{\widehat{X}} \mathsf{y}_X x$, which implies $\mathsf{S}_X\bot_{\widehat{X}} \leqslant_X x$. □

Example 2.11. Fix $\mathbb{Y}X := \{\mathsf{y}_X x \mid x \in X\}$ for any X. Then each X is $\mathbb{Y}$-complete. In particular, the poset of natural numbers is $\mathbb{Y}$-complete, but it is not directed-complete. Therefore $\mathbb{Y}$ is not admissible.

3. Intermezzo: Pataraia's construction of fixed points

A well-known theorem about partial orders states that an order-preserving map on a pointed dcpo has the least fixed point; remarkably, as shown by D. Pataraia [13], this statement has a fully constructive proof, formalizable in higher-order intuitionistic logic (in 2003 it found an entry to the compendium on continuous lattices and domains [9], where it is presented as a set of exercises). In what follows, we are going to use Pataraia's construction to simultaneously prove Banach's and Knaster-Tarski's theorems. Since we will refer to details of Pataraia's proof, we shall start with a concise description of his construction.

Recall that a map $f\colon P \to P$ on a poset P is *expanding* if $x \leqslant fx$, for all x. We say that an order-preserving expanding map is *inflationary.* A *dcpo* is a short name for "directed-complete poset". A poset is *pointed*, if it has the least element, usually denoted as $\bot$. Now, suppose f is order-preserving on a pointed dcpo P. Following Pataraia's line of thought, we look for subsets of P that (a) contain $\bot$, (b) are closed under f, and (c) are directed-complete. Clearly $Y := \{x \in P \mid x \leqslant fx\}$ is one of them. Let C be the intersection of all sets with (a)-(c). It is easy to see that C itself has properties (a)-(c). Therefore, $f\colon C \to C$ is inflationary. Now, the set $E(C)$ of all inflationary maps on C, ordered pointwise, is directed-complete, and —

this is the crux of the construction — it is itself a directed set. The reason is that for $g, h \in E(C)$, we have $g, h \leqslant g \circ h$. Hence $E(C)$ has a top element $m\colon C \to C$. Consequently, $f \circ m = m$, and thus $f(m(\bot)) = m(\bot)$, i.e. $m(\bot) \in C$ is a fixed point of f. If $x \in P$ is some other fixed point of f, then $\downarrow x := \{y \in P \mid y \leqslant x\}$ satisfies (a)-(c), hence $C \subseteq \downarrow x$ and consequently, $m(\bot) \leqslant x$.

4. Reconciliation of the theorems of Banach and Knaster-Tarski

The main result of this section is the following:

Theorem 4.1. *Fix an admissible class of ideals J. Let $T\colon X \to X$ be a non-expansive map on a J-complete distance space X. Suppose that there exists $\phi \in JX$, which is a fixed point of T^*. Then T has a fixed point, which is the least fixed point of T above $\mathsf{S}_X\phi$.*

Proof. Since X is complete, $\mathsf{S}_X\phi$ exists. Since T is non-expansive:

$$\begin{aligned}
X(\mathsf{S}_X(T^*(\phi)), T(\mathsf{S}_X\phi)) &= \widehat{X}(T^*(\phi), \mathsf{y}_X(T(\mathsf{S}_X\phi))) \\
&= \inf_{z\in X}(X(z, T(\mathsf{S}_X\phi)) \dot{-} T^*(\phi)(z)) \\
&= \inf_{z\in X}(X(z, T(\mathsf{S}_X\phi)) \dot{-} \sup_{w\in X}(\phi w + X(w, Tz))) \\
&= \inf_{w\in X}(\sup_{z\in X}(X(z, T(\mathsf{S}_X\phi)) \dot{-} X(w, Tz)) \dot{-} \phi w) \\
&= \inf_{w\in X}(X(Tw, T(\mathsf{S}_X\phi)) \dot{-} \phi w) \\
&\leqslant \inf_{w\in X}(X(w, \mathsf{S}_X\phi) \dot{-} \phi w) \\
&\leqslant 0 \dot{-} \phi(\mathsf{S}_X\phi)) \\
&= 0.
\end{aligned}$$

Hence $\mathsf{S}_X(T^*(\phi)) \leqslant_X T(\mathsf{S}_X\phi)$, and so $\mathsf{S}_X\phi \leqslant_X T(\mathsf{S}_X\phi)$. We can now follow steps of Pataraia's construction, the only difference being that instead of $\bot$, we use $\mathsf{S}_X\phi$. Thus T has a fixed point, which is of the form $m(\mathsf{S}_X\phi)$ for some $m\colon C \to C$. Recall that $C \subseteq X$ is the smallest set closed under T and directed lubs that contains $\mathsf{S}_X\phi$. Suppose now that x_1 is also fixed by T, and that $\mathsf{S}_X\phi \leqslant x_1$. Being the lower cone of a fixed point, $\downarrow x_1$ is closed under T and directed lubs. As a consequence $m(\mathsf{S}_X\phi) \in C \subseteq \downarrow x_1$. □

The most common situation when ideals are fixed points of T^* is the following:

Lemma 4.2. *Let $T\colon X \to X$ be a non-expansive map on a distance space X. If for some $x_0 \in X$ the sequence $(T^n x_0)_{n\in\omega}$ is forward Cauchy, then the associated $\mathbb{C}$-ideal ϕ is a fixed point of T^*.*

Proof. Let us first show that $\phi \leqslant_{\widehat{X}} T^*(\phi)$. Fix $N \in \omega$, $y \in X$ and choose $\varepsilon > 0$ such that $\sup_{n\geqslant N} X(y, T^n x_0) < \varepsilon$. Then there is $\delta > 0$ with $\sup_{n\geqslant N} X(y, T^n x_0) + \delta < \varepsilon$. Use Cauchyness to get $M \geqslant N$ such that for all $i \geqslant M$, $X(T^M x_0, T^i x_0) < \delta$. Hence

$\sup_{i \geqslant M} X(T^M x_0, T^i x_0) \leqslant_X \delta$. Consequently,

$$\begin{aligned} T^*(\phi)(y) &= \inf_{z \in X}(X(y, Tz) + \inf_{M \in \omega} \sup_{i \geqslant M} X(z, T^i x_0)) \\ &\leqslant_X \inf_{z \in X}(X(y, Tz) + \sup_{i \geqslant M} X(z, T^i x_0)) \\ &\leqslant_X X(y, T^{M+1} x_0) + \sup_{i \geqslant M} X(T^M x_0, T^i x_0) \\ &\leqslant_X \sup_{n \geqslant M} X(y, T^n x_0) + \delta \\ &\leqslant_X \sup_{n \geqslant N} X(y, T^n x_0) + \delta \\ &< \varepsilon. \end{aligned}$$

By the choice of ε, we conclude $T^*(\phi)(y) \leqslant_X \sup_{n \geqslant N} X(y, T^n x_0)$. This holds for any $N \in \omega$ and $y \in X$, therefore $T^*(\phi) \leqslant_{\widehat{X}} \phi$. For the converse, let $y, z \in X$, $m \in \omega$ and $n \geqslant m$. Then $X(y, T^{n+1} x_0) \leqslant_X X(y, Tz) + X(Tz, T^{n+1} x_0) \leqslant_X X(y, Tz) + X(z, T^n x_0)$ and thus $\phi y \leqslant_X X(y, Tz) + \phi z$. Consequently, $\phi \leqslant_{\widehat{X}} T^*(\phi)$. □

As we shall see Thm. 4.1 generalizes common features of three classic fixed point theorems. In fact all three theorems can be thought of as instances of Thm. 4.1 as soon as we accept that they differ by the initial choice of a point at which we start iteration of T.

Theorem 4.3 (Banach)**.** *A contraction*[1] $T : X \to X$ *on a* $\mathbb{C}$*-complete distance space* X *has a unique fixed point.*

Proof. The ideal $\phi z := \inf_{m \in \omega} \sup_{n \geqslant m}(z, T^n x_0)_{n \in \mathbb{N}}$ is a fixed point of T^*, for any choice of $x_0 \in X$. The fixed point of T, guaranteed by Thm 4.1, is unique by contractivity of T. □

Theorem 4.4 (Knaster-Tarski)**.** *A non-expansive map* $T : X \to X$ *on a* $\widehat{(\cdot)}$*-complete distance space has the least and the greatest fixed point.*

Proof. The ideal $\phi z := \inf_{m \in \omega} \sup_{n \geqslant m} X(z, T^n(\perp))$ is a fixed point of T^*. By Thm 4.1, there exists a fixed point of T, which is least above $\bigvee_{n \in \omega} T^n(\perp)$, and hence least in X.

Since T is non-expansive on X iff it is non-expansive on X^{op}, the same construction applied to X^{op} in place of X produces the least fixed point of T in X^{op}, i.e. the greatest fixed point of X. □

Theorem 4.5. *Any non-expansive map* $T : X \to X$ *on a* $\mathbb{C}$*-complete distance space with the least element* $\perp_X$ *has the least fixed point.*

Proof. The ideal $\phi z := \inf_{m \in \omega} \sup_{n \geqslant m} X(z, T^n(\perp))$ is a fixed point of T^*. By Thm 4.1, there exists a fixed point of T, which is least above $\bigvee_{n \in \omega} T^n(\perp)$, and hence least in X. □

Remark 4.6. Let X be a $\widehat{(\cdot)}$-complete distance space and T be a non-expansive map with a fixpoint. Then set $\mathrm{Fix}(T) = \{x \in X \mid x = Tx\}$ of fixed points of T is a complete lattice in the induced order.

[1]Since our distance is valued in extended reals, we define a contraction to be a function $T : X \to X$ such that there exists $0 < c < 1$ with $X(x, y) \leqslant c \cdot X(Tx, Ty) < \infty$, for all $x, y \in X$.

Indeed, let $D \subseteq \mathrm{Fix}(T)$. By Example 2.10, the least upper bound $\bigvee_X D$ exists in $(X, \leqslant_X)$. It is easy to see that $(T^n(\bigvee_X D))_{n \in \omega}$ is a forward Cauchy sequence. By Lemma 4.2 and Theorem 4.1, there exists a least fixed point of T above $\bigvee_X D$, which is clearly the least upper bound of D in $(\mathrm{Fix}(T), \leqslant_X)$.

Analogously one proves that in a $\mathbb{C}$-complete space, the set $\mathrm{Fix}(T)$ is a dcpo.

5. Reconciliation of the theorems of Caristi and Bourbaki-Witt

Recall the Caristi theorem [5]:

Theorem 5.1 (Caristi)**.** *Let X be a complete metric space and let $\varphi\colon X \to [0, \infty)$ be a lower semicontinuous function. Suppose $T\colon X \to X$ is an arbitrary mapping which satisfies $X(x, Tx) + \varphi(Tx) \leqslant \varphi(x)$ for each $x \in X$. Then T has a fixed point.*

In what follows we will show that the theorem is still valid in non-symmetric distance spaces which are $\mathbb{C}$-complete. Our main result of this section is:

Theorem 5.2. *Let X be an $\mathbb{C}$-complete distance space and let $\varphi\colon X \to [0, \infty)$ be a lower semicontinuous function. Suppose $T\colon X \to X$ is an arbitrary mapping which satisfies $X(x, Tx) + \varphi(Tx) \leqslant \varphi(x)$ for each $x \in X$. Then T has a fixed point.*

Interestingly, when X is a directed-complete poset and φ is taken to be constant, then Theorem 5.2 becomes the well-known:

Theorem 5.3 (Bourbaki-Witt)**.** *Let X be an dcpo. Suppose $T\colon X \to X$ is expanding. Then T has a fixed point.*

Therefore both the Caristi and the Bourbaki-Witt theorems are instances of our Theorem 5.2.

Proof of Theorem 5.2. The first part of our proof is based on the method proposed by Oettli and Théra in [12].

For every $x \in X$ define

$$S(x) := \{z \in X \mid X(x, z) + \varphi(z) \leqslant \varphi(x)\}$$

and

$$\delta(x) := \sup_{z \in S(x)} (\varphi(x) - \varphi(z)).$$

Then clearly $x \in S(x)$ and $0 \leqslant \delta(x) < \infty$ for any $x \in X$. Let $x_0 \in X$. Choose $x_1 \in S(x_0)$ so that

$$\delta(x_0) - 1 \leqslant \varphi(x_0) - \varphi(x_1),$$

and, inductively, for any $n \in \mathbb{N}$ such that x_n has been defined, choose $x_{n+1} \in S(x_n)$ so that

$$\delta(x_n) - \frac{1}{n+1} \leqslant \varphi(x_n) - \varphi(x_{n+1}).$$

Observe that if $y \in S(x_{k+1})$ for some $k \in \mathbb{N}$, then $X(x_{k+1}, y) \leqslant \varphi(x_{k+1}) - \varphi(y)$. However, $X(x_k, x_{k+1}) \leqslant \varphi(x_k) - \varphi(x_{k+1})$, hence $X(x_k, y) \leqslant X(x_k, x_{k+1}) + X(x_{k+1}, y) \leqslant \varphi(x_k) - \varphi(x_{k+1}) + \varphi(x_{k+1}) - \varphi(y) = \varphi(x_k) - \varphi(y)$, i.e. $y \in S(x_k)$. This demonstrates

$S(x_{k+1}) \subseteq S(x_k)$. Therefore,

$$
\begin{aligned}
0 &\leqslant \delta(x_{k+1}) \\
&= \sup_{z \in S(x_{k+1})} (\varphi(x_{k+1}) - \varphi(z)) \\
&= \sup_{z \in S(x_{k+1})} [(\varphi(x_{k+1}) - \varphi(x_k)) + (\varphi(x_k) - \varphi(z))] \\
&\leqslant \sup_{z \in S(x_k)} [(\varphi(x_{k+1}) - \varphi(x_k)) + (\varphi(x_k) - \varphi(z))] \\
&= \delta(x_k) - (\varphi(x_k) - \varphi(x_{k+1})) \\
&\leqslant \frac{1}{k+1}.
\end{aligned}
$$

Also $X(x_k, x_{k+1}) \leqslant \varphi(x_k) - \varphi(x_{k+1}) \leqslant \delta(x_k) \leqslant 1/k$, and so $(x_k)_{k\in\mathbb{N}}$ is a forward Cauchy sequence. Let $\phi := \inf_{n\in\mathbb{N}} \sup_{k\geqslant n} X(-, x_k)$ and $w = \mathsf{S}_X\phi$.

Since $0 = X(w, w) = \inf_{n\in\mathbb{N}} \sup_{k\geqslant n} X(x_k, w)$, we know that

$$\forall \eta > 0 \; \exists n \; \forall k \geqslant n \quad X(x_k, w) < \eta. \tag{5.1}$$

We claim that $w \in S(x_n)$ for all $n \in \mathbb{N}$. Suppose that for some $\varepsilon, \eta > 0$ we have $\varepsilon + \eta < X(x_n, w)$. By (5.1), there exists $k \geqslant n$ such that $X(x_n, w) \leqslant X(x_n, x_k) + X(x_k, w)$ and $X(x_k, w) < \eta$. Therefore $\varepsilon < X(x_n, x_k) \leqslant \varphi(x_n) - \varphi(x_k) \leqslant \varphi(x_n) - \varphi(x_k) + \varphi(x_k) - \lim_{n\to\infty} \varphi(x_n) \leqslant \varphi(x_n) - \varphi(w)$. This demonstrates that $X(x_n, w) + \varphi(w) \leqslant \varphi(x_n)$, as required.

As a consequence,

$$\delta(w) = 0, \tag{5.2}$$

$$\varphi(w) = \varphi(z), \text{ for all } z \in S(w), \tag{5.3}$$

$$w \leqslant_X z, \text{ for all } z \in S(w), \tag{5.4}$$

Next, we claim that $(S(w), \leqslant_X)$ is a dcpo. Let D be a directed subset of $S(w)$. Hence D is a forward Cauchy net in X. Let ψ be its associated $\mathbb{C}$-ideal. Since X is $\mathbb{C}$-complete, by Example 2.9 the poset $(X, \leqslant_X)$ is directed-complete, and, moreover, $v := \mathsf{S}_X\psi$ is a directed supremum of D. Observe that for any $d \in D$, since $d \in S(w)$, and $d \leqslant_X v$, then by (5.4) we have $w \leqslant_X v$. In addition, $\varphi(v) \leqslant \inf_{d\in D} \sup_{c\geqslant d} \varphi(c) = \varphi(w)$ follows by lower semicontinuity of φ and (5.3), respectively. Therefore $v \in S(w)$, which concludes the proof that $S(w)$ is directed-complete.

Finally, we will show that $S(w)$ is closed under the application of T. Indeed, suppose that $z \in S(w)$. Hence $X(w, Tz) \leqslant X(w, z) + X(z, Tz) \leqslant (\varphi(w) - \varphi(z)) + (\varphi(z) - \varphi(Tz)) = \varphi(w) - \varphi(Tz)$, which shows $Tz \in S(w)$.

Note that by (5.3), for all $z \in S(w)$ we have $\phi(z) = \phi(Tz)$, and hence by the hypothesis of our theorem, $X(z, Tz) \leqslant \varphi(z) - \varphi(Tz) = 0$. Therefore $z \leqslant_X Tz$, which demonstates that the map $T\colon S(w) \to S(w)$ is expanding.

To summarize, inside the distance space X we have constructed a dcpo $S(w)$, equipped with an expanding map $T\colon S(w) \to S(w)$. By the Bourbaki-Witt theorem, T has a fixed point in $S(w)$, and hence in X. $\square$

Acknowledgements

The second author is supported by grant number N206 3761 37 funded by Polish Ministry of Science and Higher Education.

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