On the interaction of topology and order in quantale-enriched categories

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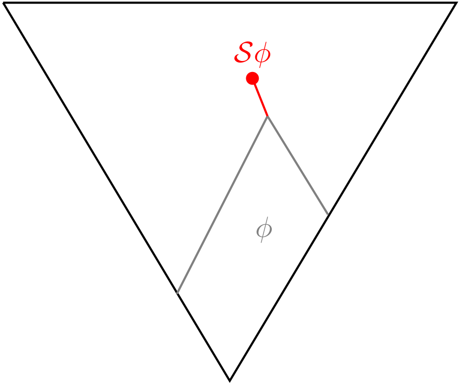
Kielce, VII.2010

A motivation from domain theory

A dcpo is a poset where every ideal (directed, lower subset) has a supremum.



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Hence a poset X is a dcpo iff the Yoneda embedding ${\downarrow}\colon X \to \mathbf{Idl}X$ has a left adjoint $\mathcal{S}\colon \mathbf{Idl}X \to X$:

$$\mathcal{S} \dashv {\downarrow}.$$

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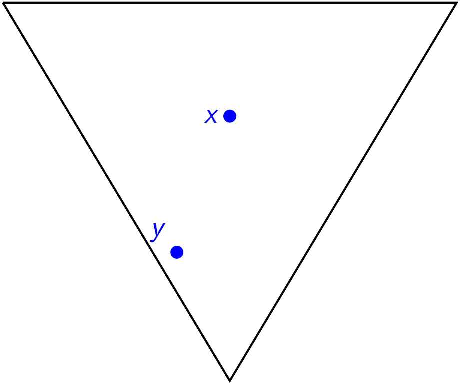

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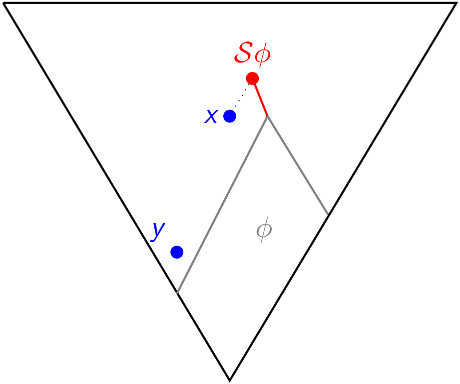

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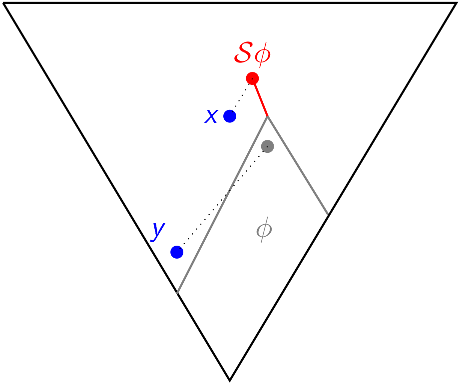

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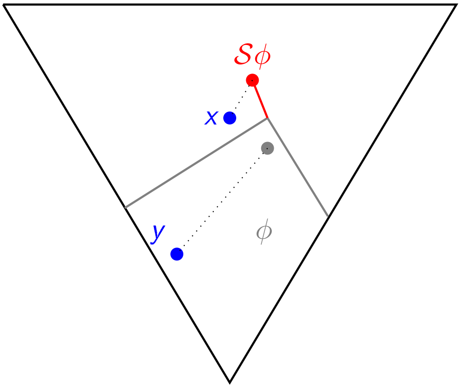

DEF.: $y \ll x$ iff $\forall \phi \; (x \leqslant \mathcal{S}\phi \iff y \in \phi)$.

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- In the lattice of opens of a locally compact Hausdorff space,
$$O \ll U \quad \text{iff} \quad O \subseteq K \subseteq U \text{ for some compact } K.$$

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J-continuous J-complete quasimetrics

We can further generalize the situation by:

- Replacing **2** by a commutative unital quantale $\mathcal{Q}$ (note: $X(-,-)\colon X \times X \to \mathcal{Q}$ satisfies quasimetric axioms!)
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DEFINITION

A quasimetric X is J-complete and J-continuous if

$$\Downarrow \dashv \mathcal{S} \dashv \downarrow$$

for

$$\Downarrow\colon X \to JX,$$

$$\mathcal{S}\colon JX \to X,$$

$$\downarrow\colon X \to JX.$$

Two simplest examples of J-continuous J-complete quasimetrics

- For $J = \mathbf{Idl}$ and $\mathcal{Q} = \mathbf{2}$, we get continuous dcpos.
- For $J = \widehat{X}$ and $\mathcal{Q} = \mathbf{2}$ we get completely distributive complete lattices.

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Now define $J = \mathbb{A}$ by:

$$\phi \in \mathbb{A}X \quad \text{iff} \quad \phi x = \inf_{i \in I} \sup_{j \geq i} X(x, x_j)$$

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EXAMPLE: For $\mathcal{Q} = ([0,\infty], +)$, $\mathbb{A}$-complete metrics are exactly complete metrics.
EXAMPLE: For $\mathcal{Q} = \mathbf{2}$, $\mathbb{A}$-complete $\mathbb{A}$-continuous quasimetrics are precisely continuous dcpos.

Three short stories about ($\mathbb{A}$-continuous) $\mathbb{A}$-complete quasimetrics

1. On embedding into continuous dcpos.
2. On a duality theorem.
3. On fixed points of non-expansive maps.

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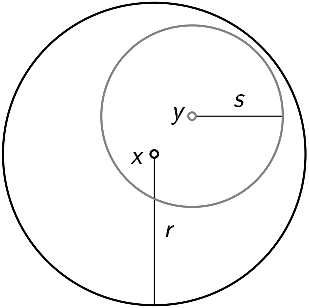
Edalat, A. and Heckmann, R. (1998) A computational model for metric spaces. *Theoretical Computer Science* **193**(1–2), pp. 53–73.



$$\mathbf{B}X := \{\langle x, r\rangle \mid x \in X \text{ and } r \geqslant 0\} \subseteq X \times \mathbb{R}_+$$

$$\langle x, r\rangle \leqslant \langle y, s\rangle \quad \text{iff} \quad X(x, y) + s \leqslant r$$

$$X \cong \{\langle x, 0\rangle \mid x \in X\} (= \max(\mathbf{B}X) \text{ providing } X \text{ is } T_1).$$

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THEOREM If X is, in addition, T_1, then the Yoneda embedding $\downarrow : X \to \max(BX)$ is a homeomorphism from the natural topology on X generated by sets of the form

$$O(x, r) := \{y \in X \mid \Downarrow(x, y) < r\}$$

to the subspace Scott topology on BX.

In fact,

$$O(x, r) = \downarrow^{-1}(\Uparrow\langle x, r\rangle).$$

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- Consider
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- If $\mathcal{Q} = \mathbf{2}$, $\mathcal{F}X$ is the collection of Scott-open filters on X.
- Define
$$\mathcal{F}(f\colon X \to Y)\colon \mathcal{F}Y \to \mathcal{F}X$$
$$\mathcal{F}(f)(\alpha) := \alpha \circ f.$$

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For $\mathcal{Q} = \mathbf{2}$ the above is known as the Lawson duality.

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- We'll show more: both are instances of a single theorem with a constructive proof.

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7. If $z \in X$ is other fixed point, then $C \subseteq {\downarrow}z$, hence $\mathit{fix}(T) \leqslant_X z$.

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Instead of conclusion

(James Caristi, 1976) Let $T\colon X \to X$ be an arbitrary map on a complete metric space. If there exists a l.s.c. map $\varphi\colon X \to [0, \infty)$ such that:

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4. Hence (Nonsymmetric Caristi) is constructive iff the Bourbaki-Witt theorem is constructive.

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3. Hence (*) iff the map $\langle x, \varphi x\rangle \mapsto \langle Tx, \varphi(Tx)\rangle$ is expanding.
4. Hence (Nonsymmetric Caristi) is constructive iff the Bourbaki-Witt theorem is constructive.
5. But Andrej Bauer proved that the Bourbaki-Witt theorem has no constructive proof whatsoever. q.e.d.